

# Describing a Combinatorics Problem with a System of Polynomial Equations

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## ABSTRACT

This paper provides a method to describe and solve a combinatorics problem using systems of polynomial equations. These systems, however, are too large to be solved by hand. The goal of this paper is to give the reader two techniques to solve these systems. The first technique uses Buchberger's Algorithm to find a Gröbner basis for the system. The second technique addresses and solves the problem if finding a Gröbner basis is computationally difficult.



## 1. INTRODUCTION

In 1970, Milton Bradley(c) created a game played on a hexagon-shaped grid called 'Drive Ya Nuts.' The game consists of seven hexagonal nuts, each having a unique arrangement of the numbers one through six on each side. The object of the game is to arrange the nuts on the grid in such a way that adjacent sides of the nuts have matching numbers. Up to rotation of the entire game board, there are possible 235; 146; 240 ways to place the nuts. Suppose that we were able to go through every combination and check if it were a solution every second, then it would take 7.46 years to find all solutions. In this paper we will determine how many of these combinations are solutions. Disregard-

ing a brute force attempt to find all solutions, we begin by describing this game by a system of polynomial equations.

## 2. DESCRIBING THE GAME

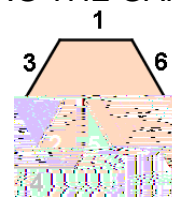


Figure 1: Nut  $B_0$  in initial rotation state

### 2.1 Notation and Description

Each nut has a particular ordering of one through six. I will refer to the ordering of a specific nut by a 6-tuple, headed by any number with subsequent numbers listed clockwise. The first entry of the 6-tuple will correspond to number located on the north side of the nut, which we call position 0. The second entry will correspond to the number located on the east-north-east side of the nut, which we call position 1. Following entries will correspond to the next side moving clockwise up to position 5 corresponding to west-north-west.

If the zero entry of the 6-tuple is 1, then we shall call that the **initial rotation state of the nut**. For instance, in Figure 1, the second entry of  $B_0$  is 2 and the fourth entry is 5. In a randomly assigned order, here are the definitions for each nut in the initial rotation state:  $B_0 = (1; 6; 2; 4; 5; 3)$ ,  $B_1 = (1; 4; 6; 2; 3; 5)$ ,  $B_2 = (1; 6; 5; 3; 2; 4)$ ,  $B_3 = (1; 4; 3; 6; 5; 2)$ ,  $B_4 = (1; 2; 3; 4; 5; 6)$ ,  $B_5 = (1; 6; 4; 2; 5; 3)$ ,  $B_6 = (1; 6; 5; 4; 3; 2)$ .

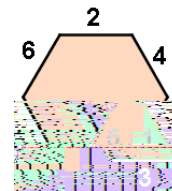


Figure 2: Nut  $B_0$  rotated 4 times

example,  $\mathbf{B}_0$  rotated four times would look like Figure 2. The corresponding 6-tuple for  $\mathbf{B}_0$  rotated four times would be (2; 4; 5; 3; 1; 6):

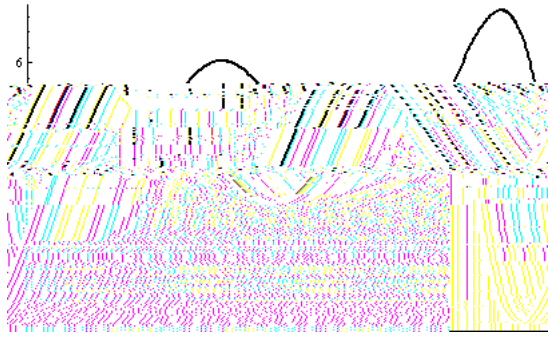


Figure 5:  $f_{0,1}(x)$  on  $[0; 5]$

### 4.3 Finding a Solution

We have 27 equations of degree 16 or less in 16 variables with rational coefficients and we want to find a solution to system. If we multiply an equation in the system by some  $K \in \mathbb{R}[g_0; g_1; g_2; g_3; g_4; g_5; g_6; p_2; p_3; p_4; p_5; a; b; c; d]$  or

**Proof.** Using a computer algebra system, the Gröbner basis of  $I(V(I))$  is  $\langle 1+12d; 1+24c; 1+120b; 1+720a; 4+p_6; 3+p_5; 3+p_4; p_3; 3+p_2; 6+g_6; 5+g_5; 4+g_4; 3+g_3; 2+g_2; 1+g_1; g_0 \rangle$ .  $\square$

Our goal was to find a Gröbner basis for  $I$ , but at best we have found a Gröbner basis that contains  $I$  and has the same variety.

**Theorem 4.3.**  $\langle I \rangle = \langle 1+12d; 1+24c; 1+120b; 1+720a; 4+p_6; 3+p_5; 3+p_4; p_3; 3+p_2; 6+g_6; 5+g_5; 4+g_4; 3+g_3; 2+g_2; 1+g_1; g_0 \rangle$

**Proof.** The proof follows directly from Lemma 7 on page 34 of [2].  $\square$

We hope in the future to be able to show that the above is an equality and show that  $\langle 1+12d; 1+24c; 1+120b; 1+720a; 4+p_6; 3+p_5; 3+p_4; p_3; 3+p_2; 6+g_6; 5+g_5; 4+g_4; 3+g_3; 2+g_2; 1+g_1; g_0 \rangle$  is a Gröbner basis for our ideal.

## 5. SOLVING CIPRA'S PROBLEM

One interesting application to this technique of describing a combinatorics problem as a system of polynomial equations

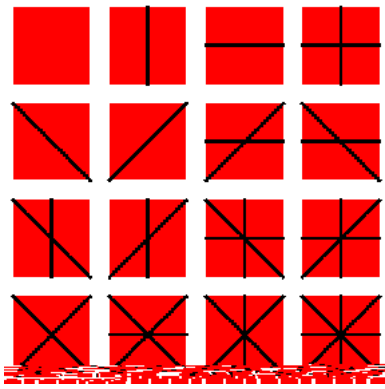


Figure 9: All of the squares in Cipra's Puzzle

these systems using ideals, varieties, and Gröbner bases, but that is dependent on the power of computing available to calculate the Gröbner bases. Lacking high-powered computing, we still are able to calculate a solution using Mathematica's Reduce function iteratively.

## 7. REFERENCES